J. Fluid Mech. (1987), vol. 180, pp. 319–326
Printed in Great Britain

Streaming from a cylinder due to an acoustic source

By N. RILEY
School of Mathematics and Physics, University of East Anglia, Norwich, NR4 7TJ, UK

(Received 22 May 1986)

An acoustic line source is placed in the neighbourhood of a circular cylinder. Acoustic streaming is generated in a Stokes shear-wave layer, at the surface of the cylinder, beyond which it persists. It is the streaming outside the Stokes layer, for large values of a streaming Reynolds number, that is the subject of the present paper.

1. Introduction

In a recent article Lighthill (1978) has re-examined the phenomenon of acoustic streaming. He demonstrates, in particular, the fundamental principle that it is the attenuation of acoustic energy flux that makes momentum flux available to force the streaming motion. He identifies two such forms of attenuation. In the first, attenuation takes place in the main body of the fluid, as for example in an ultrasonic beam, whilst in the second it takes place owing to fluid friction in the neighbourhood of a solid boundary. In this paper we are concerned with an example of the second type, as we consider the two-dimensional streaming that arises when an acoustic line source is placed close to a circular cylinder.

Studies of this kind originated with the work of Rayleigh (1884), and were continued *inter alia* by Schlichting (1932), Nyborg (1953) and Westervelt (1953). In some of these studies a solid body vibrates in a fluid at rest, rather than being itself at rest in an acoustic standing wave. As Lighthill (1978) points out the streaming in each case may be treated theoretically in the same way. The same treatment has been applied by Longuet-Higgins (1953) to the streaming that is induced when waves form on the surface of a liquid of finite depth. All of these earlier studies neglect the effect of the fluid's inertia on the steady streaming. It was Stuart (1963) who pointed out that outside a Stokes shear layer close to the boundary the flow is characterized by a streaming Reynolds number R_s, such that for $R_s \ll 1$ fluid inertia may be ignored whilst for $R_s \gg 1$ the streaming flow has a double structure. Thus there is a boundary layer, of thickness $O(aR_s^{-\frac{1}{2}})$ where a is a typical length, within which the streaming is relatively strong and outside which there is a weak potential flow. Riley (1967), reviewing this early work, showed that when $R_s = O(1)$ the Navier–Stokes equations for steady flow are required, and, in particular, that the Reynolds stresses make no direct contribution to the outer streaming, which is induced indirectly from the action of such stresses in the Stokes shear layer. In subsequent work Davidson & Riley (1972) have carried out a theoretical and experimental study of the boundary layers and jets that form on, and in the neighbourhood of, a vibrating cylinder when $R_s \gg 1$. An experimental study by Bertelsen (1974), also for $R_s \gg 1$, concentrates on the flow in the boundary layer. Riley (1975), using higher-order boundary-layer theory, attempts to reconcile the measured boundary-layer profiles of Bertelsen with those

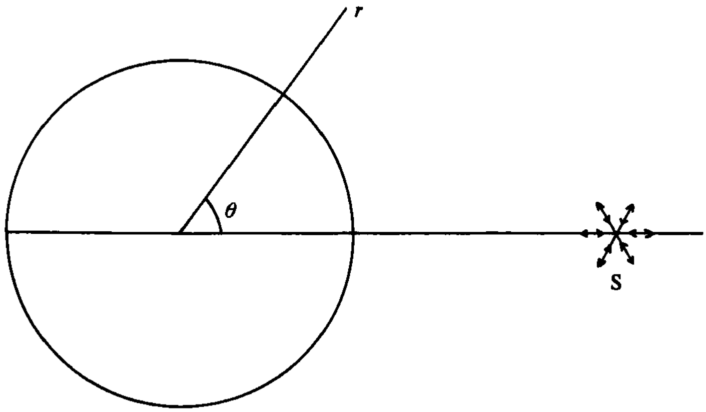

FIGURE 1. The physical configuration and coordinate system. The source is located at S.

predicted theoretically. Further such attempts have been made by Duck & Smith (1979) and Haddon & Riley (1979) who solve the Navier–Stokes equations in a bounded annular region for finite values of R_s.

All of the above studies possess a high degree of symmetry, namely about each of two perpendicular axes. Wang (1972), in a more asymmetric situation considers the streaming that is induced when an acoustic line source is placed close, and parallel, to a circular cylinder. He analyses the flow both in the Stokes layer and outside it, although he restricts his attention to $R_s \ll 1$. But as Lighthill (1978) points out, all really noticeable acoustic streaming motions are associated with $R_s \gg 1$. Wang (1982, 1984) has also considered the streaming induced when an acoustic source is placed close to a sphere or a plane boundary. In the present paper we address ourselves to the configuration of Wang (1972), shown in figure 1, when R_s is large.

As we have already indicated the parameter R_s is essentially a Reynolds number associated with the acoustic streaming motion outside the Stokes shear-wave layer. For the case under consideration it is defined as follows. If m is the strength of the acoustic source which has frequency ω, and ν is the kinematic viscosity of the fluid, then $R_s = m^2/4\pi^2a^2\omega\nu$, where a is the radius of the cylinder. A second parameter associated with the flows under consideration is $\epsilon = m/2\pi\omega a$, and $\epsilon \ll 1$. The theoretical development of Riley (1967), which is also applicable here, is for $\epsilon \ll 1$, $R_s = O(1)$. Since we particularly wish to concentrate on the case $R_s \gg 1$ we note that the theory is *formally* valid in the limits $\epsilon \to 0$, $R_s \to \infty$, where the limits are taken in that order. For the flow outside the Stokes shear-wave layer Riley (1967) expands the dimensionless stream function as

$$\psi = \sum_{n=0}^{\infty} \epsilon^n \psi_n(r, \theta, t; R_s), \tag{1.1}$$

where $m/2\pi a$, ω^{-1} and a have been chosen as a typical velocity, time and length respectively; the coordinates (r, θ) are defined in figure 1. We now substitute (1.1) into the unsteady Navier–Stokes equations for incompressible flow. These equations will be appropriate for $r \ll c/\omega$ where c is the speed of sound in the fluid. At first order we have

$$\psi_0 = \psi_{0s}(r, \theta) \cos t, \tag{1.2}$$

where $\psi_{0s}(r, \theta)$ represents the steady irrotational flow from a source in the neighbourhood of a circular cylinder. This inviscid solution yields a velocity of slip $U_s(\theta)$

at the surface $r = 1$, which is adjusted in the Stokes shear layer. If the term $O(\epsilon)$ in (1.1) is decomposed as

$$\psi_1 = \psi_1^{(u)}(r, \theta, t; R_s) + \psi_1^{(s)}(r, \theta; R_s), \tag{1.3}$$

then Riley shows that it is only when the equation for ψ_3 is considered that the equation for $\psi_1^{(s)}$ emerges. That equation is the full Navier–Stokes equation for steady flow at Reynolds number R_s. Matching with the solution in the Stokes shear layer shows that $\partial\psi_1^{(s)}/\partial r = -\frac{3}{4}U_s\,\mathrm{d}U_s/\mathrm{d}\theta$ at $r = 1$. This latter result was essentially noted by Rayleigh (1884).

It is the full Navier–Stokes equations that are our starting point for the discussion of the acoustic streaming in §2. We make the boundary-layer approximation appropriate to $R_s \gg 1$, and consider the solution within, and beyond, a boundary layer of thickness $O(R_s^{-\frac{1}{2}})$. The flow is symmetrical about the lines $\theta = 0, \pi$, and the boundary layers that form above and below this line of symmetry collide so that jet-like flows erupt from $r = 1, \theta = 0, \pi$. These jets have different strengths, and make a contribution to the force balance that tends to repel the cylinder from the acoustic source. This asymmetrical feature of the outer flow is further emphasized, in the flow beyond the boundary layer, by the streamline pattern of the mean flow.

2. The solution for $R_s \gg 1$

With an acoustic line source placed at a distance R from the centre of a circular cylinder, as in figure 1, the slip velocity that is induced at the surface of the cylinder, if the fluid is assumed to be inviscid, may be written as

$$u_s(R, \theta, t) = U_s(R, \theta) \cos t, \tag{2.1}$$

where

$$U_s(R, \theta) = \frac{2R \sin\theta}{R^2 + 1 - 2R\cos\theta}. \tag{2.2}$$

The adjustment that is necessary to satisfy the no-slip condition at the surface of the cylinder takes place in a Stokes layer of thickness $O(\epsilon/R_s^{\frac{1}{2}})$. Furthermore the Reynolds stresses that act within the Stokes layer induce a steady motion, or streaming $O(\epsilon)$. The details of the flow structure within the Stokes layer are well-documented, as for example by Stuart (1963). The streaming motion persists to the edge of the Stokes layer where it takes the value $\epsilon u_e(R, \theta)$ with

$$u_e = -\tfrac{3}{4}U_s\frac{\mathrm{d}U_s}{\mathrm{d}\theta} = \frac{3}{2}\left\{\frac{4R^3\sin\theta - R^2(R^2+1)\sin 2\theta}{(R^2+1-2R\cos\theta)^3}\right\}, \tag{2.3}$$

and use has been made of (2.2). This 'edge' velocity is, in turn, responsible for driving the streaming motion outside the Stokes layer. It has been shown by Riley (1967) that, for a situation of the type under consideration, the Reynolds stresses do not directly contribute to this outer steady streaming which is governed by the steady Navier–Stokes equations with R_s as Reynolds number. Thus we have, with (r, θ) as the cylindrical polar coordinates of figure 1, and (u, v) the corresponding components of velocity,

$$\frac{\partial u}{\partial\theta} + \frac{\partial(rv)}{\partial r} = 0, \tag{2.4}$$

$$\frac{u}{r}\frac{\partial u}{\partial\theta} + v\frac{\partial u}{\partial r} + \frac{uv}{r} = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \frac{1}{R_s}\left(\nabla^2 u + \frac{2}{r^2}\frac{\partial v}{\partial\theta} - \frac{u}{r^2}\right), \tag{2.5}$$

$$\frac{u}{r}\frac{\partial v}{\partial\theta} + v\frac{\partial v}{\partial r} - \frac{u^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{R_s}\left(\nabla^2 v - \frac{v}{r^2} - \frac{2}{r^2}\frac{\partial u}{\partial\theta}\right), \tag{2.6}$$

where $\nabla^2 = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\theta^2$, and the velocity components are related to $\psi_1^{(s)}$ by $u = \partial\psi_1^{(s)}/\partial r$, $rv = -\partial\psi_1^{(s)}/\partial\theta$. The boundary conditions require that $(u, v, p) \to 0$ as $r \to \infty$ and that $u = u_e$, $v = 0$ at $r = 1$. The latter conditions are effectively a matching condition between the outer flow and the Stokes layers.

For $R_s \ll 1$ Wang (1972) has addressed the problem of this outer streaming. Here we concentrate on the situation for $R_s \gg 1$, and we find it appropriate to expand the solution of (2.4)–(2.6) as

$$\left.\begin{aligned} u &= u_1 + R_s^{-\frac{1}{2}} u_2 + \ldots, \\ v &= R_s^{-\frac{1}{2}} v_1 + \ldots, \\ p &= R_s^{-\frac{1}{2}} p_2 + \ldots . \end{aligned}\right\} \tag{2.7}$$

The expansions (2.7) reflect the boundary-layer nature of the outer streaming, for which it is appropriate to introduce the boundary-layer coordinate

$$r = 1 + R_s^{-\frac{1}{2}} \bar{y}. \tag{2.8}$$

2.1. *The outer boundary layer at first order*

Introducing (2.7), (2.8) into (2.4) to (2.6) gives the following boundary-layer problem at leading order:

$$\frac{\partial u_1}{\partial\theta} + \frac{\partial v_1}{\partial \bar{y}} = 0, \tag{2.9a}$$

$$u_1 \frac{\partial u_1}{\partial\theta} + v_1 \frac{\partial u_1}{\partial \bar{y}} = \frac{\partial^2 u_1}{\partial \bar{y}^2}, \tag{2.9b}$$

with

$$\left.\begin{aligned} v_1 = 0, \quad u_1 = u_e \quad &\text{on} \quad \bar{y} = 0, \\ u_1 \to 0 \quad &\text{as} \quad \bar{y} \to \infty. \end{aligned}\right\} \tag{2.10}$$

With u_e given by (2.3) the solution of (2.9), (2.10) has to be completed using numerical methods. The method that we have adopted is one that is described in detail by Davidson & Riley (1972), for a similar problem to that under consideration here. For the parabolic equation (2.9b) the numerical solution must commence at a stagnation point of attachment of the outer flow. Such stagnation points are located at $r = 1$, $\theta = \pm\theta_s$, where θ_s is determined from $u_e(R, \theta_s) = 0$. The value of $\theta_s = \theta_s(R)$ so determined is shown in figure 2. For $\theta > \theta_s$, where $u_e > 0$, we introduce into (2.9) the variable $\bar{\theta} = \theta - \theta_s$, whilst for $\theta < \theta_s$, where now $u_e < 0$, we introduce the variables $\tilde{\theta} = \theta_s - \theta$, $\tilde{u}_1 = -u_1$, in order to complete the numerical solution which is, of course, symmetrical about the line $\theta = 0$. A feature of the solution is that the momentum flux in the boundary layer is non-zero as $\bar{\theta} \to \pi - \theta_s$, and as $\tilde{\theta} \to \theta_s$. The momentum flux is given by

$$M(\theta) = \int_0^\infty u_1^2 \, \mathrm{d}\bar{y}, \tag{2.11}$$

and we define $M_1 = M(0)$, $M_2 = M(\pi)$. The ratio M_1/M_2 is shown in figure 2 for various values of R. Figure 2 clearly demonstrates the asymmetry, about $\theta = \frac{1}{2}\pi$, of the flow. As the source location approaches the cylinder the stagnation points of attachment associated with the outer steady streaming approach $\theta = 0$, and the momentum flux ratio increases dramatically. Owing to symmetry there will be boundary-layer collisions at $\theta = 0$, π, with jets emerging along those directions.

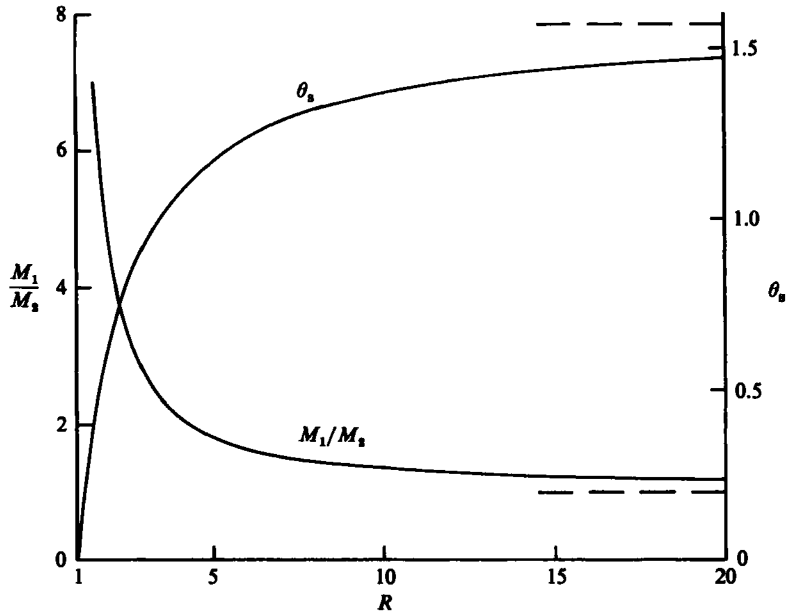

FIGURE 2. The stagnation point of attachment θ_s, and jet momentum flux ratio M_1/M_2 as functions of R. In each case the broken line represents the limit as $R \rightarrow \infty$.

Lighthill (1978) has emphasized that such jets are a characteristic feature of acoustic streaming whether by attenuation of acoustic energy flux in the main body of the fluid or, as in the present case, by friction in the neighbourhood of a solid boundary.

For the potential-flow problem there is a force of attraction between the source and the cylinder. Although this will dominate in the present case we note that the flux of momentum in the jets across a large contour which contains the cylinder makes a contribution $\epsilon^2(M_1 - M_2)/R_s^{\frac{1}{2}}$ to a repulsive force. This is unaffected by the inviscid outer flow at $O(\epsilon/R_s^{\frac{1}{2}})$ that we now consider.

2.2. *The inviscid flow beyond the boundary layer*

Beyond the boundary layer of the outer flow that has been discussed above, the flow velocities are $O(\epsilon/R_s^{\frac{1}{2}})$, the flow behaves as if the fluid were inviscid and the fluid motion is induced by entrainment into the boundary layer, and into the jets that form along $\theta = 0, \pi$. When the entrainment velocities are known the outer inviscid flow is that due to a given source distribution along $\theta = 0, \pi$ for $r > 1$ and $r = 1$, $0 \leqslant \theta \leqslant 2\pi$, and may be easily calculated using complex-variable methods.

The entrainment velocity $v_{1\infty} = v_1(\bar{y} = \infty)$ is obtained during the course of the boundary-layer calculation, and is shown in figure 3 for various values of R. Consider next entrainment into the jets. In the course of their experimental programme Davidson & Riley (1972) observed, in a not dissimilar investigation to the present one, that the jet profile had achieved a similarity form within one cylinder diameter from the point at which the jet is formed. The similarity solution proposed by Bickley (1937) for the plane jet may be expressed as

$$\left.\begin{aligned} \psi_1^{(s)} &= \{9N(x+x_0)\}^{\frac{1}{3}} \tanh \eta, \\ \eta &= \left\{\frac{N}{24(x+x_0)^2}\right\}^{\frac{1}{3}} \bar{y}, \end{aligned}\right\} \tag{2.12}$$

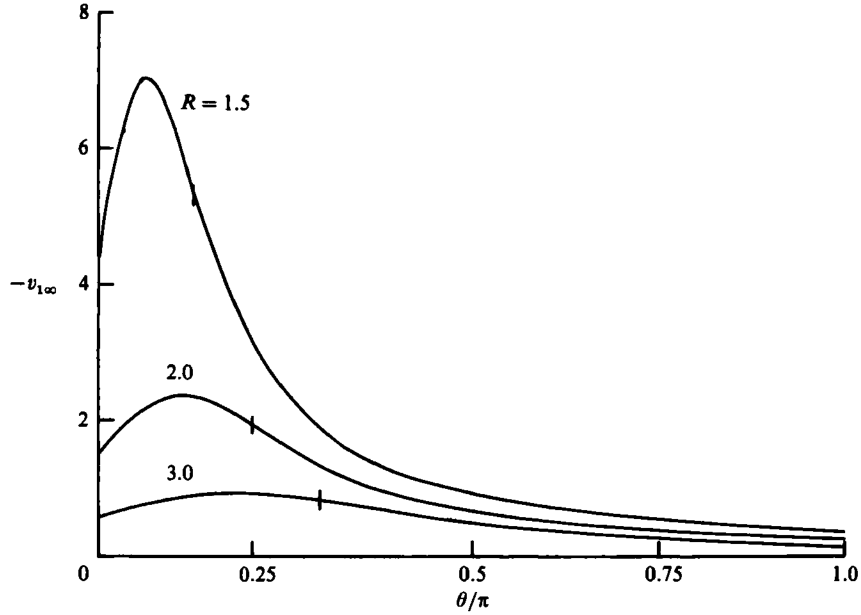

FIGURE 3. The entrainment velocity into the boundary layer for various values of R. The stagnation point θ_s is denoted in each case thus —+—.

where $2N$ is the invariant momentum flux in the jet, $(x, \bar{y})$ are rectangular coordinates with x measured along the line $\theta = 0$, and the constant x_0 may be interpreted as a virtual origin for the similarity solution. An expression analogous to (2.12) describes the jet flow along $\theta = \pi$. From (2.12) we find that along the jet

$$v_{1\infty} = -(\tfrac{1}{3}N)^{\frac{1}{3}}(x+x_0)^{-\frac{2}{3}}. \tag{2.13}$$

The constants N and x_0 remain to be determined. In the boundary-layer collision region Davidson & Riley (1972) have argued, as have Stewartson (1958) and Lyne (1971) in similar situations but different physical contexts, that the flow is essentially inviscid, and that the velocity profiles at the end of each boundary layer are convected around to emerge essentially unchanged. In that case, for the jet along $\theta = 0$ we have $N = M_1$, and along $\theta = \pi$, $N = M_2$. With only the two free parameters, N and x_0, it is not possible to describe the flow in the jets completely for $|x| > 1$. To fix x_0 we may, for example, insist on continuity of mass flow from the boundary layers to the jet, with a discontinuity in $v_{1\infty}$, or we may accept a discontinuity in mass flow by taking $v_{1\infty}$ to be continuous. The calculations that we have carried out show only small differences in the solutions between either of these choices; for the results presented below we have chosen x_0 by equating the mass flux in the jet at $x = 1$ with that in the boundary layers impinging at $\theta = 0$. A similar choice has been made for the jet along $\theta = \pi$.

We are now in a position to determine the solution outside the outer boundary layer. First we transform the plane $z = re^{i\theta} = x + iy$ to the complex Z-plane, where

$$Z = z + z^{-1} = X + iY, \tag{2.14}$$

so that the circle $r = 1$ is transformed to the slit $|X| \leqslant 2$, $Y = 0$ and the axis $|x| > 1$, $y = 0$ to the slit $|X| > 2$, $Y = 0$. The velocity components with which we are concerned are denoted in this transformed plane by $(R_s^{-\frac{1}{2}}U_2\, R_s^{-\frac{1}{2}}V_1)$. The source strength

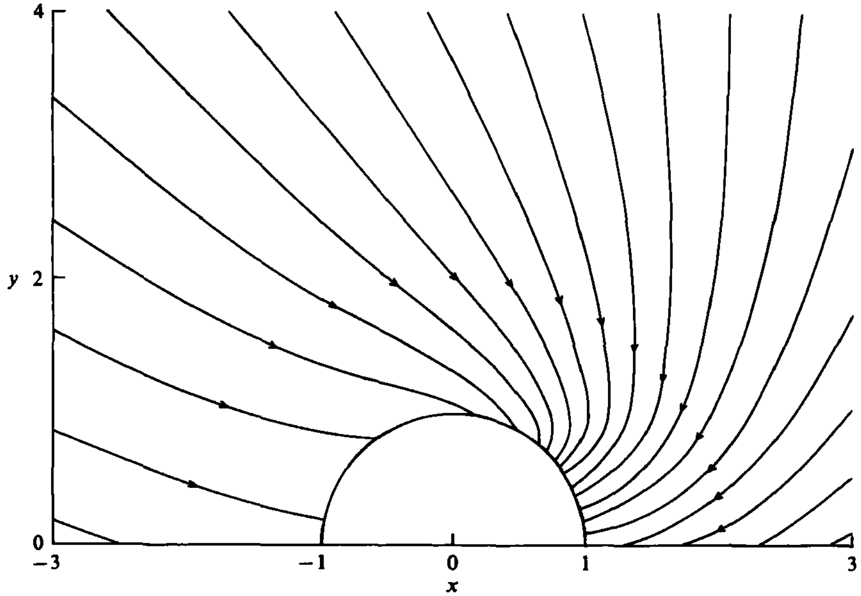

FIGURE 4. The outer-flow mean streamlines for $R = 1.5$. The streamlines are plotted at equal intervals $\delta\Psi = \frac{1}{18}(\Psi_1 - \Psi_2)$ where Ψ_1 is the value of Ψ at $x = 3.0$, $y = 0.1$ and Ψ_2 the value at $(-3.0, 0.1)$.

along the axis $Y = 0$ is given by $q(X) = 2V_1(X, 0)$ so that the stream function Ψ associated with (U_2, V_1) may be written as

$$\Psi(X, Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(s)\tan^{-1}\left(\frac{Y}{X-s}\right)\mathrm{d}s, \tag{2.15}$$

and we have

$$q(X) = \begin{cases} v_{1\infty}\dfrac{(X^2-4)^{\frac{1}{2}}-X}{(X^2-4)^{\frac{1}{2}}}, & X < -2, \\ \dfrac{2v_{1\infty}}{(4-X^2)^{\frac{1}{2}}}, & |X| < 2, \\ v_{1\infty}\dfrac{(X^2-4)^{\frac{1}{2}}+X}{(X^2-4)^{\frac{1}{2}}}, & X > 2. \end{cases} \tag{2.16}$$

The mean streamlines for the outer inviscid flow, calculated from $\Psi(X, Y) = \text{constant}$, are shown in figure 4 for $R = 1.5$. The highly asymmetric nature of the flow is made evident from this streamline pattern which reflects the higher entrainment velocities that are to be found, both in the boundary layer and the jet, on the source-side of the cylinder.

3. Conclusions

In this paper we have calculated, at leading order, the acoustic streaming that is induced when an acoustic line source is placed in the neighbourhood of, and parallel to, a circular cylinder, for values of the streaming Reynolds number $R_s \gg 1$. From the asymmetric nature of the steady jet-like flows which erupt from the surface of the cylinder we deduce that for large R_s there is a contribution to the force balance, from the acoustic streaming, that tends to repel the cylinder from the source. The streamline pattern of the outer steady flow further emphasizes the asymmetry of the streaming.

REFERENCES

BERTELSEN, A. F. 1974 *J. Fluid Mech.* **64**, 589.

BICKLEY, W. G. 1937 *Phil. Mag.* **23**, 727.

DAVIDSON, B. J. & RILEY, N. 1972 *J. Fluid Mech.* **53**, 287.

DUCK, P. W. & SMITH, F. T. 1979 *J. Fluid Mech.* **91**, 93.

HADDON, E. W. & RILEY, N. 1979 *Q. J. Mech. Appl. Maths* **32**, 265.

LIGHTHILL, M. J. 1978 *J. Sound Vib.* **61**, 391.

LONGUET-HIGGINS, M. S. 1953 *Phil. Trans R. Soc. Lond.* A**245**, 535.

LYNE, W. H. 1971 *J. Fluid Mech.* **45**, 13.

NYBORG, W. L. 1953 *J. Acoust. Soc. Am.* **25**, 68.

RAYLEIGH, LORD 1884 *Phil. Trans R. Soc. Lond.* A**175**, 1.

RILEY, N. 1967 *J. Inst. Maths Applics* **3**, 419.

RILEY, N. 1975 *J. Fluid Mech.* **68**, 801.

SCHLICHTING, H. 1932 *Phys. Z.* **33**, 327.

STEWARTSON, K. 1958 In *Proc. Symp. Boundary Layer Research Freiburg i Br. 1957*, pp. 57–91. Springer.

STUART, J. T. 1963 *Laminar Boundary Layers*, chap. 7. Oxford University Press.

WANG, C-Y. 1972 *Recent Research on Unsteady Boundary Layers, IUTAM Symposium Quebec 1971*, p. 1653. Quebec: Laval University Press.

WANG, C-Y. 1982 *J. Acoust. Soc. Am.* **71**, 580.

WANG, C-Y. 1984 *J. Acoust. Soc. Am.* **75**, 108.

WESTERVELT, P. J. 1953 *J. Acoust. Soc. Am.* **25**, 60.